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STABILIZATION OF THE POSITION OF A CIRCULAR MEMBRANE†

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A uniform circular membrane, the edges of which are clamped to a large, absolutely rigid ring, is considered. In its undeformed state, the membrane lies in the same plane as the ring. One-dimensional displacements of the ring and membrane in a direction perpendicular to that plane are investigated. The problem of stabilizing the state of the system by means of an external control force operating in that direction on the ring is studied. Stabilizing control is realized by linear feedback with respect to the displacement and velocity of the ring, and the integral of the displacement and deformations of the membrane. Allowance is made for a delay in the control loop. The stability of the control process is considered, and the regions of asymptotic stability of the desired position of equilibrium of the membrane in the space of the feedback coefficients are found.

The analogous problem of a rod which is deformed by bending [1–3] or subject to longitudinal and torsional deformation [4] has already been considered. The problem of controlling the motion of a membrane has been investigated in a different formulation (see [5], for example). The problem examined below may be of interest to the problem of stabilizing space film structures [6].

1. THE EQUATIONS OF MOTION

Consider a uniform circular membrane with surface density ρ. The edge of the membrane is clamped to an absolutely solid ring of mass M and radius a. In its undisturbed state the surface of the membrane lies in the same plane as the ring. The membrane tension σ is assumed to be the same at every point of the deformed and undeformed surface and to operate in a tangential plane.

We will use a polar system of coordinates r, θ in the plane of the ring with origin at its centre. The equations of motion of the system and the boundary condition can be written in the form

$$\rho\left(\frac{\partial^2 u}{\partial t^2}+\frac{d^2 z}{dt^2}\right)=\sigma\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right] \tag{1.1}$$

$$M\frac{d^2 z}{dt^2}=F-\sigma\int_0^{2\pi}\left.\frac{\partial u}{\partial r}\right|_{r=a}a\,d\theta \tag{1.2}$$

$$u(a,\theta,t)=0 \tag{1.3}$$

Here $u(r, \theta, t)$ is the deviation of the membrane at time t from its undisturbed plane surface in a direction perpendicular to that surface (when there is no deformation $u(r, \theta, t) \equiv 0$), z is the deviation (displacement of the ring from the desired (given) position in a direction perpendicular to its plane, and F is the control force, with vector perpendicular to the plane of the ring. Equation (1.1) describes the vibrations of the membrane [7, 8] under the given acceleration of the ring. Equation (1.2) describes the motion of the ring and the second term on the right-hand side describes the force on the membrane from the direction of the ring.

We will introduce the new variable

$$v(r,\theta,t)=u(r,\theta,t)+z(t) \tag{1.4}$$

which characterizes the total deviation of the deformed membrane from its desired position, and the dimensionless variables, denoted by asterisks, with the formulae

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$$u = au^*,\quad v = av^*,\quad z = az^*,\quad r = ar^*,\quad t = \tau t^* \left(\tau^2 = \frac{\rho a^2}{\sigma}\right) \tag{1.5}$$

Substituting relations (1.4) and (1.5) into Eqs (1.1) and (1.2) and omitting the asterisks, we obtain

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \tag{1.6}$$

$$\mu\left.\frac{\partial^2 v}{\partial t^2}\right|_{r=1} = f - \frac{1}{2\pi}\int_0^{2\pi}\left.\frac{\partial v}{\partial r}\right|_{r=1} d\theta \quad \left(\mu = \frac{M}{2\rho\pi a^2},\; f = \frac{F}{2\pi\sigma a}\right) \tag{1.7}$$

In relation (1.7) μ and f are the dimensionless mass of the ring and the force applied to it. This relation, which is obtained from Eq. (1.2) using the equation

$$v(1,\theta,t) = z(t) \tag{1.8}$$

is taken as the boundary condition in the new boundary-value problem. Equation (1.8) is a consequence of boundary condition (1.3). If the control force f is independent of the variable z, the latter is cyclical.

2. THE CONTROL AND STATEMENT OF THE PROBLEM

If $f \equiv 0$, the boundary-value problem (1.6), (1.7) has the solution

$$v(r,\theta,t) = C\,(u(r,\theta,t) = 0,\; z = C)$$

where C is an arbitrary constant. This solution describes a ring which has an undeformed membrane and deviates from the desired position by an amount $z = C$. When $C = 0$ we have

$$v(r,\theta,t) = 0\,(u(r,\theta,t) = 0,\; z = 0) \tag{2.1}$$

It is interesting to consider the synthesis of a control f for which solution (2.1) of system (1.6), (1.7) is asymptotically stable. There are grounds for assuming, however, that it is impossible, simply by controlling the motion of the ring, to stabilize the position of the membrane and suppress any asymmetric elastic vibrations of the membrane. We will therefore consider the case in which the vibrations are symmetric, and the deformation of the membrane u is independent of the angle θ. In that case Eqs (1.6) and (1.7) take the form

$$\ddot{v}(r,t) = \frac{1}{r}\left(rv'(r,t)\right)',\quad \mu\ddot{v}(1,t) = f - v'(1,t) \tag{2.2}$$

The dot denotes differentiation with respect to time and the prime denotes differentiation with respect to the coordinate r. Equations (2.1) can be rewritten as follows:

$$v(r,t) = 0 \quad (u(r,t) = 0,\; z = 0) \tag{2.3}$$

We will construct a control which stabilizes the equilibrium (2.3) of system (2.2) in linear feedback form

$$T\dot{f}(t) + f(t) = -\gamma_0 v(1,t) - \gamma_1\dot{v}(1,t) - \gamma_2\int_0^t v(1,\zeta)\,d\zeta - \sum\sigma_n v''(r_n,t) \tag{2.4}$$

Here $T > 0$ is a dimensionless time constant characterizing the delay in the control loop γ_0, γ_1 and γ_2 are constant feedback coefficients with respect to the displacement z of the ring, its derivative and integral, and σ_n is a feedback coefficient (constant) with respect to the deformation of the membrane when $r = r_n$. Everywhere here and below $n = 1, \ldots, N$ and summation is carried out from $n = 1$ to $n = N$. The control process starts at time $t = 0$. In order to realize the feedback (2.4), a positional pickup, together velocity and deformation pickup (tensometers) are needed.

The linear boundary-value problem (2.2), (2.4) has a spectrum of eigenvalues λ. We will specify the problem of stabilizing solution (2.3) of system (2.2) as follows. In the space of the feedback coefficients (2.4), it is required to find a region of values for which all the eigenvalues λ are such that $\operatorname{Re}\lambda < 0$.

For comparison we will also consider a system with feedback (2.4) consisting of a ring of mass M and an undeformed membrane of mass $\rho\pi a^2$. In the same dimensionless variables the equations of motion of this system, which basically consists of just one absolutely rigid body of mass $M + \rho\pi a^2$, have the form

$$\left(\mu+\frac{1}{2}\right)\ddot{z}=f,\quad T\dot{f}+f=-\gamma_0 z-\gamma_1\dot{z}-\gamma_2\int_0^t z(\zeta)d\zeta \tag{2.5}$$

3. THE CHARACTERISTIC EQUATION

We will seek a solution of boundary-value problem (2.2), (2.4) in the form

$$v(r,t)=Ke^{\lambda t}R(r)$$

where K is a constant, λ is an eigenvalue and $R(r)$ is an eigenfunction.

For the function $R(r)$ we obtain the boundary-value problem

$$\lambda^2 rR(r)=\left[rR'(r)\right]' \tag{3.1}$$

$$\left[\mu\lambda^2R(1)+R'(1)\right](T\lambda+1)\lambda+\left(\gamma_0\lambda+\gamma_1\lambda^2+\gamma_2\right)R(1)+\lambda\sum\sigma_nR''(r_n)=0 \tag{3.2}$$

Introducing the new variable $y = r\lambda$, we can, as we know [7, 8], represent the function $R(r)$, which depends not only on the radius r but also on the parameter λ, in the form $R(y)$. Relations (3.1) and (3.2) can then be written as follows (the subscript y denotes differentiation with respect to the argument y):

$$R(y)=R_{yy}(y)+\frac{1}{y}R_y(y) \tag{3.3}$$

$$\begin{aligned}\Delta(\lambda)=&\left[\mu\lambda^3(T\lambda+1)+\gamma_0\lambda+\gamma_1\lambda^2+\gamma_2\right]R(\lambda)+\\&+\lambda^2(T\lambda+1)R_y(\lambda)+\lambda^3\sum\sigma_nR_{yy}(r_n\lambda)=0\end{aligned} \tag{3.4}$$

The solution of the Bessel equation (3.3) with $R(0) = 1$, $R'(0) = 0$ can be given in the form of a series with positive coefficients [8]

$$R(y)=1+\left(\frac{y}{2}\right)^2+\frac{1}{2!^2}\left(\frac{y}{2}\right)^4+\ldots+\frac{1}{k!^2}\left(\frac{y}{2}\right)^{2k}+\ldots \tag{3.5}$$

Equation (3.4) is the required characteristic equation of the system. Since relations (3.3) and (3.4) are homogeneous with respect to the function R, the condition $R(0) = 1$ does not limit the generality of the analysis. In the case of a fixed ring, the characteristic quasi-polynomial $R(\lambda)$ of the system is described by expression (3.5). When $\gamma_2 = 0$, both sides of Eq. (3.2) as well as (3.4) must be divided by λ.

It follows from relations (3.4) and (3.5) that $\Delta(0) = \gamma_2$. As $\lambda \to \infty$, $R(\lambda) \to \infty$ and $\Delta(\lambda) \to \infty$. Thus for $\gamma_2 < 0$ Eq. (3.4) has a real root $\lambda > 0$, and solution (2.3) is unstable. Let $\Delta_1(\lambda) = \Delta(\lambda)/\lambda$. Then when $\gamma_2 = 0$, $\Delta_1(0) = \gamma_0$. When $\lambda \to \infty$, $\Delta_1(\lambda) \to \infty$. Thus when $\gamma_2 = 0$, $\gamma_0 < 0$ (positive feedback) Eq.(3.4) has a real root $\lambda > 0$ and equilibrium (2.3) is unstable. Let $\Delta_2(\lambda) = \Delta_1(\lambda)/\lambda$. Then when $\gamma_2 = \gamma_0 = 0$, $\Delta_2(0) = \gamma_1$. As $\lambda \to \infty$, $\Delta_2(\lambda) \to \infty$. Thus when $\gamma_2 = \gamma_0 = 0$, $\gamma_1 < 0$ (negative damping) Eq. (3.4) has a real root $\lambda > 0$ and equilibrium (2.3) is unstable. The inequalities $\gamma_2 > 0$, $\gamma_0 > 0$ (for $\gamma_2 = 0$), $\gamma_1 > 0$ (for $\gamma_2 = \gamma_0 = 0$) are necessary conditions of asymptotic stability of solution (2.3). Necessary and sufficient conditions for stability (the exact regions of stability in the space of the feedback coefficients) will be obtained below using the method of D-partitions [9].

In Eq. (3.4) we put $\lambda = i\omega$, where ω is a real number, and consider its real and imaginary parts

$$\left[T\mu\omega^4 - \gamma_1\omega^2 + \gamma_2\right]J_0(\omega) + T\omega^3 J_1(\omega) = 0$$
$$(\gamma_0 - \mu\omega^2)\omega J_0(\omega) - \omega^2 J_1(\omega) - \omega^3 \sum \sigma_n R_{yy}(r_n i\omega) = 0 \tag{3.6}$$

Here $J_0(\omega) = R(i\omega)$, $J_1(\omega) = -iR_y(i\omega)$ are Bessel functions of the first kind of order zero and one, respectively.

In the space of the system parameters, relations (3.6) define the image of the imaginary axis $\lambda = i\omega$, $-\infty < \omega < +\infty$. They are unchanged when ω is replaced by $-\omega$. We will therefore find the boundary of the region of asymptotic stability in the parameter space by constructing the surface (3.6) for $0 \leqslant \omega < \infty$.

4. THE REGIONS OF STABILITY

We will construct the regions of asymptotic stability analytically, starting from particular cases and then proceeding to more general ones.

Suppose first that

$$\sigma_n = 0, \ \gamma_2 = 0, \ T = 0 \tag{4.1}$$

If $\gamma_2 = 0$, we must first of all divide both sides of Eqs (3.6) by ω. Instead of (3.6) under conditions (4.1) we obtain the equations

$$\gamma_1\omega J_0(\omega) = 0 \quad (\gamma_0 - \mu\omega^2)J_0(\omega) - \omega J_1(\omega) = 0 \tag{4.2}$$

When $\omega = 0$ we obtain the equation $\gamma_0 = 0$ from (4.2). As we know [8], the zeros of the functions $J_0(\omega)$, $J_1(\omega)$ are different and moreover alternate. Thus, when $\omega > 0$ Eqs (4.2) are only valid if $\gamma_1 = 0$. For $\omega > 0$ the second of Eqs (4.2) can be written in the form

$$\gamma_0 = \mu\omega^2 + \zeta(\omega), \quad \zeta(\omega) = \omega J_1(\omega) / J_0(\omega) \tag{4.3}$$

Let ω_k ($k = 1, 2, \ldots$) denote the positive zeros of the Bessel function $J_0(\omega)$, $J_1(\omega)$, numbered in increasing order. From the differential relations between the functions $J_0(\omega)$ and $J_1(\omega)$ we can see that the derivative of the function $\zeta(\omega)$ is always positive. Thus, the quantity (4.3) increases strictly monotonely from zero to $+\infty$ as ω changes from zero to ω_1, and increases strictly monotonely from $-\infty$ to $+\infty$ as ω increases from ω_k to ω_{k+1} ($k = 1, 2, \ldots$). Thus, as ω changes from zero to infinity the point (4.3) traverses the axis $\gamma_1 = 0$ from $-\infty$ to $+\infty$ an infinite number of times. Hence the boundaries of the region of asymptotic stability, if there is one, are on the straight lines $\gamma_0 = 0$ and $\gamma_1 = 0$. It follows from Section 3 that outside the region $\gamma_0 \geqslant 0$, $\gamma_1 \geqslant 0$, for the roots λ of Eq. (3.4), Re $\lambda > 0$. Thus, under condition (4.1), the region of asymptotic stability, if it exists, can be described by the inequalities

$$\gamma_0 > 0, \ \gamma_1 > 0 \tag{4.4}$$

It turns out that there is asymptotic stability in the region (4.4), which we denote by D.

This can be proved by the method described in [1, 2, 4, 10, 11]. We multiply both sides of Eq. (3.1) by the conjugate function $\bar{R}(r)$ and integrate from zero to one. Using boundary conditions (3.2) and the condition $R'(0) = 0$, we obtain

$$\lambda^2\left[\int_0^1 rR(r)\bar{R}(r)dr + \mu R(1)\bar{R}(1)\right] + \lambda\gamma_1 R(1)\bar{R}(1) +$$
$$+\gamma_0 R(1)\bar{R}(1) + \int_0^1 rR'(r)\bar{R}'(r)dr = 0$$

In domain (4.4) the coefficients of the resulting quadratic equation in λ are non-negative. Thus for all its eigenvalues λ, Re$\lambda \leqslant 0$. Since Re $\lambda = 0$ only on the boundary of region (4.4), inside that region for all the eigenvalues λ, Re $\lambda < 0$.

In the case when $\gamma_2 = T = 0$ also, the region of asymptotic stability of the equilibrium state $z(t) \equiv 0$ of system (2.5) can be described by inequalities (4.4). Thus the "pliability" of the membrane has no influence on the region of stability in this case.

Suppose now that $T > 0$. We will consider the more general case than (4.1) where

$$\sigma_n = 0, \quad \gamma_2 = 0 \tag{4.5}$$

Dividing Eq. (3.6) by ω and assuming first that $\omega = 0$, and then $\omega > 0$, we find that the boundary of the region of stability is made up of segments of the straight line $\gamma_0 = 0$ and the straight line

$$\gamma_0 = \mu\omega^2 + \zeta(\omega), \quad \gamma_1 = \mu T\omega^2 + T\zeta(\omega) \ (0 \leq \omega < \infty) \tag{4.6}$$

The parametric equations (4.6) describe a straight line, since they imply that

$$\gamma_1 = T\gamma_0 \tag{4.7}$$

As the quantity ω changes from zero to $+\infty$, the point (4.6) traverses the entire straight line (4.7) an infinite number of times.

Let $D(T)$ (Fig. 1) denote the region defined by the inequalities

$$\gamma_0 > 0, \quad \gamma_1 > T\gamma_0 \tag{4.8}$$

For $\gamma_0, \gamma_1 \in D(T)$, for all the eigenvalues λ, Re $\lambda \neq 0$ since Re $\lambda = 0$ only when $\gamma_0 = 0$ or $\gamma_1 = T\gamma_0$. We will construct the set $D(T)$ in the space of the three parameters γ_0, γ_1, T for $0 < T < \infty$. The eigenvalues λ are a continuous function of T. As $T \to 0$ we have $D(T) \to D$, where D is the region of asymptotic stability (4.4) in case (4.1). It follows that in case (4.5) solution (2.3) is asymptotically stable if and only if $\gamma_0, \gamma_1 \in D(T)$.

Using the Hurwitz conditions we find that the region of asymptotic stability of equilibrium $z(t) \equiv 0$ of system (2.5) in the case when $\gamma_2 = 0$ can also be described by inequalities (4.8). Hence, as in the case $\gamma_2 = T = 0$, the "pliability" of the membrane has no influence on the region of stability.

Suppose now that $T > 0$ and $\gamma_2 \neq 0$. We will consider a more general case than (4.5), where only

$$\sigma_n = 0 \ (n = 1, 2, \ldots, N) \tag{4.9}$$

It was shown in Section 3 that when $\gamma_2 < 0$ solution (2.3) of system (2.2), (2.4) is unstable. We shall therefore assume that $\gamma_2 > 0$. We choose some quantity $\gamma_2 > 0$ and construct the region of stability $D(T, \gamma_2)$ in the plane of the coefficients γ_0, γ_1. Obviously the boundary $\Gamma(T, \gamma_2)$ of the region $D(T, \gamma_2)$ is described by parametric equations similar to Eqs (4.6) but with the additional term γ_2/ω^2 in the second of them; moreover in this case $0 < \omega < \omega_1$. As $\omega \to 0$ the curve $\Gamma(T, \gamma_2)$ gets closer and closer to the axis $\gamma_0 = 0$, and as $\omega \to \omega_1$ gets closer and closer to the straight line

$$\gamma_1 = T\gamma_0 + \gamma_2 / \omega_1^2$$

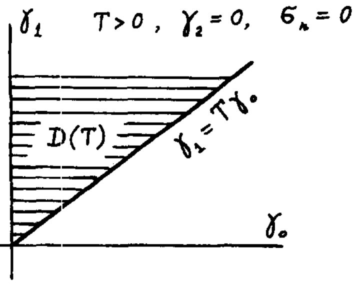

Fig. 1.

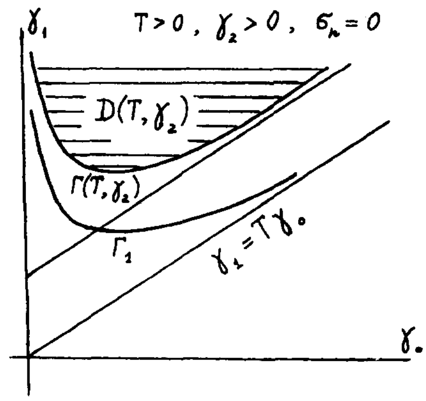

Fig. 2.

which is parallel to the straight line (4.7) and lies above it.

Let the region of asymptotic stability of equilibrium $z(t) \equiv 0$ of system (2.5) be denoted by $E(T, \gamma_2)$. Using the Hurwitz conditions we find that the region $E(T, \gamma_2)$ is bounded by a branch of the hyperbola

$$\gamma_0(\gamma_1 - T\gamma_0) - \left(\mu + \frac{1}{2}\right)\gamma_2 = 0 \quad (\gamma_0 > 0,\ \gamma_1 > 0,\ \gamma_2 > 0) \tag{4.10}$$

The asymptotes of this hyperbola are the axis $\gamma_0 = 0$ and the straight line (4.7).

We will substitute the functions $\gamma_0(\omega)$ and $\gamma_1(\omega)$ of the equation of the boundary $\Gamma(T, \gamma_2)$ into the left-hand side of Eq. (4.10). Using the differential relations between Bessel functions, it can be shown that the resulting function is positive for $0 < \omega < \omega_1$. As $\omega \to 0$ it tends to zero and as $\omega \to \omega_1$ it tends to infinity. Thus the boundary $\Gamma(T, \gamma_2)$ lies above the boundary Γ_1 (4.10), an d the domain $D(T, \gamma_2)$ lies entirely inside the domain $E(T, \gamma_2)$ (Fig. 2). Thus, unlike cases (4.1) and (4.5), in case (4.9) the region of stability is smaller for a pliable membrane than for an absolutely rigid membrane.

5. THE REGION OF STABILITY WHEN TENSOMETERS ARE USED

We will consider a more general case than (4.9), where

$$\sigma_1 \neq 0,\ r_1 = 0,\ \sigma_2 = \sigma_3 = \ldots = \sigma_n = 0 \tag{5.1}$$

The last term on the left-hand side of Eq. (3.4) then has the form $\lambda^3\sigma_1/2$.

Let $\Delta_3(\lambda) = \Delta(\lambda)/\lambda^3$. Then for $\gamma_0 = \gamma_1 = \gamma_2 = 0$, $\Delta_3(0) = \mu + (\sigma_1 + 1)/2$. For $\lambda \to \infty$, $\Delta_3(\lambda) \to \infty$. Thus for $\sigma_1 < -(1 + 2\mu)$ at the point $\gamma_0 = \gamma_1 = \gamma_2 = 0$ Eq. (3.4) has a real root $\lambda > 0$, and equilibrium (2.3) is unstable. The same applies for $\sigma_1 < -(1 + 2\mu)$ and in a small neighbourhood of the point $\gamma_0 = \gamma_1 = \gamma_2 = 0$.

Suppose that, in addition to condition (5.1), the condition

$$\gamma_2 = 0 \tag{5.2}$$

applies at first.

Let $D(T, \sigma_1)$ denote a set in the half-plane $\gamma_0, \gamma_1 > 0$ bounded by the semi-axis $\gamma_0 = 0, \gamma_1 > 0$ and the curve $\Gamma(T, \sigma_1)$, the parametric equations (parametric $\omega \subset (0, \omega_1)$) of which, for constant values of $T > 0, \sigma_1$, are like Eqs (4.6) but with the additional term $\sigma_1\omega^2/(2J_0(\omega))$ in the first equation. Since $\gamma_1(\omega) > 0$ and $\gamma_1(\omega) \to \infty$ as $\omega \to \omega_1$, the curve $\Gamma(T, \sigma_1)$ lies in the upper half-plane of the plane (γ_0, γ_1) and is unbounded. Moreover, the function $\gamma_1(\omega)$ increases strictly monotonely as ω increases from 0 to ω_1, and so the curve $\Gamma(T, \sigma_1)$ has no self-intersections.

As $\sigma_1 \to 0$, $D(T, \sigma_1) \to D(T)$, since the curve $\Gamma(T, \sigma_1)$ approaches the straight line (4.7) as $\sigma_1 \to 0$. We recall that $D(T)$ is the region of asymptotic stability (4.8) in case (4.5). Thus if the coefficients σ_1 are near zero, the set $D(T, \sigma_1)$, and that set only, will be a region of asymptotic stability of the system under conditions (5.1) and (5.2). For $\sigma_1 > 0$ the quantity $\gamma_0(\omega)$ in the equation of the curve $\Gamma(T, \sigma_1)$ increases strictly monotonely from zero to infinity as ω increases from zero to ω_1, and the curve $\Gamma(T, \sigma_1)$ lies below the straight line (4.7). As σ_1 increases the curve $\Gamma(T, \sigma_1)$, remaining in the first quadrant, "gets lower and lower", approaching the semi-axis $\gamma_0 > 0, \gamma_1 = 0$. As the coefficient σ_1 increases from zero to infinity, the set $D(T, \sigma_1)$ (Fig. 3), remaining the region of stability, increases continuously and, as $\sigma_1 \to \infty$, approaches the region D (4.4), which is the region of stability both for an absolutely rigid and for an elastic membrane in case (4.1). Thus, the region of asymptotic stability can be expanded by including signals concerning the deformation in the feedback. Although the control loop has a delay T, that region can "nearly" be expanded to the domain D in which there is stability when $T = 0$.

It was shown above that the region of stability of the system under control (2.4) belongs to the half-plane $\gamma_0 > 0$. If $\gamma_0 = 0, \gamma_1 < 0$, the characteristic equation has an eigenvalue $\lambda > 0$. The curve $\Gamma(T, \sigma_1)$ lies in the upper half-plane of the plane (γ_0, γ_1). It follows from the above that the region of stability, if there is one, belongs to the first quadrant, that is, region D (4.4), whatever the values σ_n, r_n ($n = 1, 2, \ldots$). In other words, D is the largest possible region of stability.

If $\sigma_1 < -(1 + 2\mu)$, the curve $\Gamma(T, \sigma_1)$, starting at $\omega = 0$ from the point $\gamma_0 = \gamma_1 = 0$, lies, as can be shown, entirely in the second quadrant. Since when $\gamma_2 = 0$ for values γ_0, γ_1 near zero, Eq. (3.4) has a root $\lambda > 0$, the system is unstable at all points γ_0, γ_1. It follows that a necessary condition for stability of equilibrium (2.3) is the inequality

$$\sigma_1 > -(1+2\mu) \tag{5.3}$$

Tables of Bessel functions [8] show that $(1 + 2\mu) > 2J_1(\omega_1)/\omega_1$. Thus a quantity σ_1 which satisfies the inequality

$$-\sigma_* < \sigma_1 < 0, \quad \sigma_* = 2J_1(\omega_1)/\omega_1 \tag{5.4}$$

also satisfies inequality (5.3). It can be shown that under condition (5.4) the quantity $\gamma_0(\omega)$ in the equation of the curve $\Gamma(T, \sigma_1)$ increases strictly monotonely as ω increases from zero to ω_1. Moreover it is bounded if $\sigma_1 = -\sigma_*$, and unbounded otherwise. Thus under condition (5.4) the region D (T, σ_1) is unbounded, and also $D(T, \sigma_1) \subset D(T)$ (Fig. 4).

If

$$-(1+2\mu) < \sigma_1 \leqslant -\sigma_* \tag{5.5}$$

the function $\gamma_0(\omega)$ is positive for small values of ω and vanishes once in the interval $(0, \omega_1)$, and $\gamma_0(\omega) \to -\infty$ as $\omega \to \omega_1$. Thus under condition (5.5) the region of stability D (T, σ_1) in the first quadrant is bounded (Fig. 5). As σ_1 decreases from zero to $-(1 + 2\mu)$, the region of stability $D(T, \sigma_1)$ decreases continuously and strictly monotonely and contracts to the empty set.

Since under control (2.4) the region of stability of the system is in the maximum possible region D and D $(T, \sigma_1) \to D$ as $\sigma_1 \to \infty$, it is impossible to obtain a larger region of stability by putting tensometers at other points in addition to the centre of the membrane.

It is usual to introduce an integral term into the feedback in order to eliminate the static error in the control system, where $\gamma_2 \neq 0$. We have already shown that stability can only occur if $\gamma_2 > 0$. In that case, the equations of the boundary of the region of stability differ from those given in this section in that the second of the equations of type (4.6) contains the additional term γ_2/ω^2. These equations

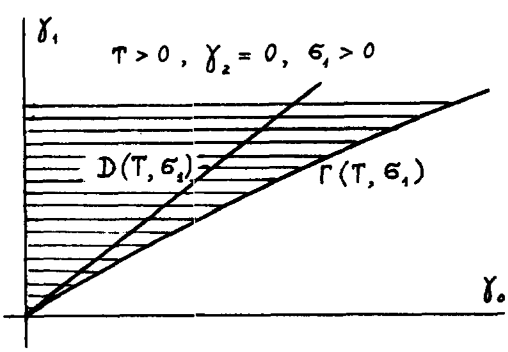

Fig. 3.

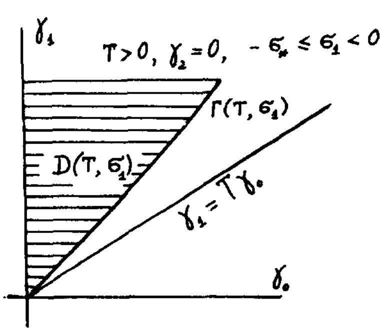

Fig. 4.

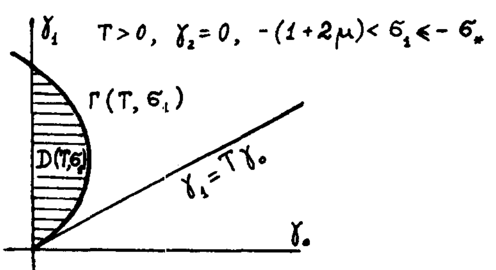

Fig. 5.

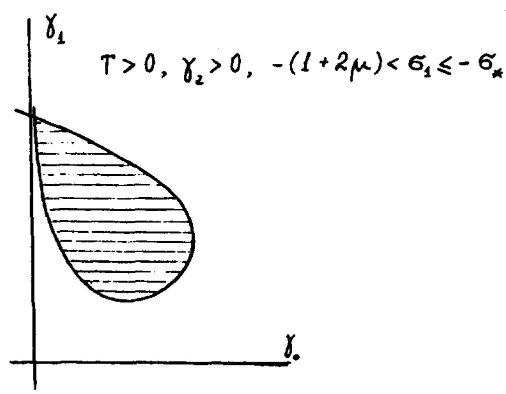

Fig. 6.

can be used to determine the structure of the region of stability. When $\sigma_1 > 0$, and also under condition (5.4), the region is unbounded and similar to that shown in Fig. 2, while under condition (5.5) it is bounded and is "drop-shaped" (Fig. 6). Inequality (5.3) is also a necessary condition for stability when $\gamma_2 \neq 0$.

The equations of the boundaries of the regions of stability obtained above can be used for the numerical construction of those regions for specific values of the system parameters.

It is interesting to see that the boundaries of the regions of stability have qualitatively the same structure as in [1–4] even though we have been considering an elastic membrane, while the problem in [1–4] concerned an elastic rod. Naturally the formulae for the boundaries of the region of stability will be different for a membrane and for a rod.

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